CS 548: Computer Vision
Image Transformation: Introduction to the Fourier Transform
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Introduction

- So far, we’ve talked about the **Spatial Domain**.

- Now we will discuss transformations that will transform the image from the spatial domain to the **Frequency Domain**.

- Specifically, we’ll talk about the **Fourier Transform**.
Why work in the Frequency Domain?

- Changes in the spatial domain $\rightarrow$ map to $\rightarrow$ values in the frequency domains

- Tasks related to frequency domain
  - Regular Filtering
    - Can apply regular filters in frequency domain:
      \[ f(x, y) \otimes h(x, y) \Leftrightarrow H(u, v)F(u, v) \]
      - Especially as filters get larger $\rightarrow$ faster to do 2 Fourier transforms and a filter multiply than convolution
  - Noise removal
    - Apply selective filters in frequency domain
      - Bandreject/bandpass filters $\rightarrow$ reject/pass specific bands of frequencies
      - Notch filters $=$ process small regions of frequency rectangle
  - Compression
    - E.g., DCT transform
Filtering Example

Original

Spectrum

Frequency Domain Filter

Image After Filtering
Noise Filtering Example

**FIGURE 4.64**
(a) Sampled newspaper image showing a moiré pattern.
(b) Spectrum.
(c) Butterworth notch reject filter multiplied by the Fourier transform.
(d) Filtered image.
Another Noise Filtering Example

**FIGURE 4.65**
(a) $674 \times 674$ image of the Saturn rings showing nearly periodic interference.
(b) Spectrum: The bursts of energy in the vertical axis near the origin correspond to the interference pattern.
(c) A vertical notch reject filter.
(d) Result of filtering. The thin black border in (c) was added for clarity; it is not part of the data. (Original image courtesy of Dr. Robert A. West, NASA/JPL.)
INTRODUCTION
TO THE
FOURIER
TRANSFORM
Background and History

• Jean Baptist Joseph Fourier
  ◦ Writes memoir in 1807 (and book in 1822) that states:
    ◦ Any periodic function can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by different coefficients
  ◦ Sum → called Fourier Series
FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier’s idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.
Application to Non-Periodic Functions

- If function not periodic (but area under curve is finite) → still express as integral of sines/cosines multiplied by weighing function

- This is called the **Fourier Transform**
Fourier Domain and Back Again

- Can transform data to **Fourier Domain** and then back to the original domain **without losing any information**
FFT: Fast Fourier Transform

- Discrete Fourier Transform algorithm
- Developed in early 1960s
- Allowed for much faster processing:
  - $O(N^2) \rightarrow O(N \log N)$
COMPLEX NUMBERS: A REVIEW
Complex Numbers

- A complex number $C$ is defined as:
  \[ C = R + jI \]

- where:
  - $R$ and $I =$ real numbers
  - $j =$ imaginary number equal to $\sqrt{-1}$

- $R \rightarrow \text{“real part”}$
- $I \rightarrow \text{“imaginary part”}$
Conjugate of Complex Numbers

- The conjugate of a complex number $C^*$ is defined as:
  
  $$C^* = R - jI$$

  - Basically, negate the imaginary part

- Product of complex number and conjugate $\rightarrow$ real number

  $$CC^* = (R + jI)(R - jI)$$
  $$= R^2 + jIR - jIR - j^2I^2$$
  $$= R^2 - (-1)I^2$$
  $$= R^2 + I^2$$
Complex Numbers as Coordinates

- One can think of complex numbers as being the coordinates on a 2D plane.

$C = R + jI = (R, I)$
Complex Numbers as Polar Coordinates

- You can also represent complex numbers in polar coordinates:

\[ C = |C|(\cos \theta + j \sin \theta) \]

- where:

\[ |C| = \sqrt{R^2 + I^2} \]
Euler’s Formula

• Euler’s formula is given by the following:

\[ e^{j\theta} = \cos \theta + j \sin \theta \]

• ASIDE: If you set \( \theta = \pi \), you get:

\[ e^{j\pi} + 1 = 0 \]
Polar Coordinate Representation with Euler’s Formula

- Substituting Euler’s Formula:

\[ e^{j\theta} = \cos \theta + j \sin \theta \]

- into the polar coordinate representation of complex numbers:

\[ C = |C| (\cos \theta + j \sin \theta) \]

- gives us:

\[ C = |C| e^{j\theta} \]
Complex Functions

A complex function $F(u)$ can be represented by:

$$F(u) = R(u) + jI(u)$$
FOURIER SERIES
Function $f(t)$

- Assume we have a function $f(t)$
  - of continuous variable $t$
  - that is periodic with period $T$ (repeats every $T$ distance)
Fourier Series

- Can express $f(t)$ → weighted sum of sines/cosines

\[ f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T} t} \]

- where the coefficients/weights are given by:

\[ c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi n}{T} t} \, dt \quad \text{for } n = 0, \pm 1, \pm 2, \ldots \]
Fourier Series: Where are the Cosines/Sines?

- If you compare this to Euler’s formula from before:

\[ e^{j\theta} = \cos \theta + j \sin \theta \]

- You can see that:

\[
f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t} = \sum_{n=-\infty}^{\infty} c_n[\cos(\frac{2\pi n}{T}t) + j \sin(\frac{2\pi n}{T}t)]
\]

- And that:

\[
\theta = \frac{2\pi n}{T} t = n^*(2\pi)^* \frac{t}{T}
\]

- So:

\[
\frac{t}{T} \rightarrow \text{where we are in the periodic function}
\]

\[
n^*(2\pi) \rightarrow \text{sets frequency of sin/cos wave}
\]
HOW WE ARE GOING TO DERIVE THE FOURIER TRANSFORM ON DISCRETE FUNCTIONS
Problem

- We will have the Fourier transform for continuous functions
  - ...which as we’ll see is: \[ \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ft} dt \]

- HOWEVER, we’re dealing with image data
  - Sampled data \(\rightarrow\) discrete coordinates \(\rightarrow\) NOT continuous!

- So, we need to find a way to derive a discrete version of the Fourier transform...
  - Note that we will be concentrating on 1D functions; later, we will extend this to 2D
Plan of Attack

- Talk about impulses and sifting property
  - We will use an impulse train to define a discrete sampled function (like image data!)
- Define the Fourier Transform itself on continuous space
- Derive the Fourier Transform on impulse train
- Define convolution in terms of the Fourier Transform
- Define discrete sampled function as product of $f(t)$ and impulse train
- Get Fourier transform of sampled function using convolution
- Finally, derive the Discrete Fourier Transform
IMPULSES AND THE SIFTING PROPERTY
A unit impulse of a continuous variable $t$ located at $t = 0$ is defined as:

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

constrained by:

$$\int_{-\infty}^{\infty} \delta(t) \, dt = 1$$

Basically, spike of infinite amplitude and zero duration, having unit area.
Impulses have the **sifting property** with respect to integration:

\[ \int_{-\infty}^{\infty} f(t) \delta(t) \, dt = f(0) \]

- **Sifting** – gives us value of function \( f(t) \) at location of impulse (here, \( t = 0 \))

- *Intuitively*: only one spot where \( \delta(t) \) has unit area (\( t = 0 \)), so that area is scaled by the value of \( f(t = 0) \)
Sifting Property Generalized

- Given an arbitrary point $t_0$, we have:

$$\int_{-\infty}^{\infty} f(t) \delta(t - t_0) \, dt = f(t_0)$$

- Basically, $t_0$ is the new center.
Unit Discrete Impulse

- **Unit discrete impulse**
  with discrete variable $x$:

$$
\delta(x) = \begin{cases} 
1 & \text{if } x = 0 \\
0 & \text{if } x \neq 0
\end{cases}
$$

- which clearly hold to this constraint:

$$
\sum_{x=-\infty}^{\infty} \delta(x) = 1
$$
Discrete Impulse Sifting Property

- Shifting property for discrete variables:
  \[ \sum_{x=-\infty}^{\infty} f(x) \delta(x-x_0) = f(x_0) \]

- Center is at \( x_0 \)
Impulse Train

- **Impulse train** – sum of infinitely many periodic impulses $\Delta T$ units apart:

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

- Can be continuous or discrete
FOURIER TRANSFORM
Fourier Transform

- Fourier transform of continuous function $f(t)$ of continuous variable $t$:

$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} \, dt$$

- where $\mu \rightarrow$ continuous variable

- Variable $t$ gets integrated out, so can also represent it this way:

$$\mathcal{F}\{f(t)\} = F(\mu)$$

- $t$ is in the spatial domain, while $\mu$ is in the frequency domain
Inverse Fourier Transform

\[ f(t) = \mathcal{Z}^{-1}\{F(\mu)\} \]

- Or, in full:

\[ f(t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t} d\mu \]
Fourier Transform Pair

- Fourier Transform Pair - the forward and inverse Fourier Transforms

Forward: \[ F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi \mu t} \, dt \]

Inverse: \[ f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi \mu t} \, d\mu \]
Frequency Domain

- After performing Fourier Transform, only variable left is $\mu \rightarrow$ frequency

- Domain of Fourier Transform is the frequency domain
  
  ◦ Also, the Fourier transform returns complex numbers!
FOURIER TRANSFORM ON IMPULSES
Fourier Transform on Unit Impulse (t = 0)

- Fourier transform on impulse at t = 0:

\[ F(\mu) = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi\mu t} dt \]

\[ = \int_{-\infty}^{\infty} e^{-j2\pi\mu t} \delta(t)dt \]

\[ = e^{-j2\pi\mu 0} = e^{0} = 1 \] (Sifting property*)

So, Fourier transform of impulse at origin \( \rightarrow \) constant

*Sifting property
\[ \int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0) \]
Fourier Transform on Unit Impulse \((t = t_0)\)

- Fourier transform on impulse at \(t = t_0\):

\[
F(\mu) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi \mu t} \, dt
\]

\[
= \int_{-\infty}^{\infty} e^{-j2\pi \mu t} \delta(t - t_0) \, dt
\]

\[
= e^{-j2\pi \mu t_0} \quad \text{(Sifting property\*)}
\]

\[
= \cos(2\pi \mu t_0) - j \sin(2\pi \mu t_0) \quad \text{(Euler’s Formula\*)}
\]

- Formula of a unit circle centered on origin in complex plane

\[
e^{j\theta} = \cos \theta + j \sin \theta
\]

\[
\int_{-\infty}^{\infty} f(t) \delta(t - t_0) \, dt = f(t_0) \quad \text{(*Sifting property)}
\]
Fourier Transform on Impulse Train

- Not as straightforward as on an impulse
  ◦ There’s the summation part to worry about
- That said, let’s derive it…
Observation

- Only real difference between forward and inverse transform is the sign of $e$:

\[ F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} \, dt \]

\[ f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} \, d\mu \]

- $f(t)$ has Fourier transform $F(\mu)$
- **Symmetry property**: $F(t)$ has transform $f(-\mu)$

\[ f(-\mu) = \int_{-\infty}^{\infty} F(t) e^{(j2\pi)(-\mu)t} \, dt = \int_{-\infty}^{\infty} F(t) e^{-j2\pi\mu t} \, dt \]
Symmetry Property: Proof

- **Proof**: we know the inverse of \( F(\mu) \) is \( f(t) \), so if we plug in \((-t)\) we get:

\[
f(-t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi \mu(-t)} d\mu
\]

- If we swap \( \mu \) for \( t \):

\[
f(-\mu) = \int_{-\infty}^{\infty} F(t)e^{-j2\pi \mu t} dt
\]

- This is the same as saying that the forward transform of \( F(t) \) is \( f(-\mu) \)
Symmetry Property

- So, since (showed this before): \( \mathcal{I}\{\delta(t - t_0)\} = e^{-j2\pi\mu t_0} \)

- Therefore: \( \mathcal{I}\{e^{-j2\pi t_0}\} = \delta(-\mu - t_0) \)

- If we set \(-t_0 = a\), then:

  \[
  \mathcal{I}\{e^{-j2\pi t_0}\} = \mathcal{I}\{e^{j2\pi t}\} = \delta(-\mu + a) = \delta(\mu - a)
  \]

- Note: last part is true because \(\delta\) is zero only when \(\mu = a\)
Impulse Train Revisited

- The impulse train is periodic:

\[ s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T) \]

- So, we can use the Fourier Series to express it (just replace \( T \) with \( \Delta T \))

\[ s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t} \]
Impulse Train as Fourier Series

- If we look at the coefficients $c_n$ for the previous series:

$$c_n = \frac{1}{\Delta T} \int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t)e^{-j\frac{2\pi n}{\Delta T} t} \, dt$$

- This integral only covers the impulse at the origin. Therefore:

$$c_n = \frac{1}{\Delta T} \int_{-\Delta T/2}^{\Delta T/2} \delta(t)e^{-j\frac{2\pi n}{\Delta T} t} \, dt$$

$$= \frac{1}{\Delta T} e^0$$

$$= \frac{1}{\Delta T}$$
Fourier Series of Impulse Train Simplified

- So, since all our coefficients equal this:
  \[ c_n = \frac{1}{\Delta T} \]

- Our Fourier Series becomes:
  \[ s_{\Delta T}(t) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T} t} \]

- Ultimately, though, we’re looking for the Fourier Transform of the Impulse train…
Almost to Fourier Transform of Impulse Train

- Summation is linear, so \((\text{Fourier Transform of sum}) = (\text{sum of Fourier Transforms of individual components})\)

- Each component = \(e^{\frac{j2\pi nt}{\Delta T}}\)

- Fourier Transform of one component:

\[
\mathcal{F}\left\{e^{\frac{j2\pi nt}{\Delta T}}\right\} = \delta\left(\mu - \frac{n}{\Delta T}\right)
\]

\[
a = \frac{n}{\Delta T}
\]

Above is true before of symmetry property from before

\[
\mathcal{F}\{e^{-j2\pi at}\} = \mathcal{F}\{e^{j2\pi at}\} = \delta(-\mu + a) = \delta(\mu - a)
\]
Fourier Transform of Impulse Train

- $S(\mu)$, the Fourier Transform of the periodic impulse train is:

$$S(\mu) = \mathcal{F}\{s_{\Delta T}(t)\}$$

$$= \mathcal{F}\left\{ \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} e^{j \frac{2\pi n}{\Delta T} t} \right\}$$

$$= \frac{1}{\Delta T} \mathcal{F}\left\{ \sum_{n=-\infty}^{\infty} e^{j \frac{2\pi n}{\Delta T} t} \right\}$$

$$= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

- …which also means the Fourier Transform of an impulse train with period $\Delta T$ is also an impulse train with period $1/\Delta T$. 
CONVOLUTION REVISITED
Convolution of Continuous Functions

- Given two continuous functions \( f(t) \) and \( h(t) \), convolution of these functions is performed with integration:

\[
f(t) \otimes h(t) = \int_{-\infty}^{\infty} f(v)h(t-v)dv
\]

- For now, we’ll use \( \otimes \) for convolution
- \( f(t) \) is the convolution mask
- \( h(t) \) is the “image”
- \( t \) indicates where the “mask” is on the image
- Also note this is the 1D case
Fourier Transform of Convolution

Let’s find the Fourier Transform of Convolution (bear with me…)

\[ \mathcal{F}\{f(t) \otimes h(t)\} = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(v)h(t - v)dv \right] e^{-j2\pi \mu t} dt \]

\[ = \int_{-\infty}^{\infty} f(v) \left[ \int_{-\infty}^{\infty} h(t - v)e^{-j2\pi \mu t} dt \right] dv \]
Fourier Transform of a Translated Function

- If we have some function \( h(t) \) translated by \( v \) \( \rightarrow \) the Fourier transform is:

\[
\mathcal{F}\{h(t - v)\} = \int_{-\infty}^{\infty} h(t - v) e^{-j2\pi\mu t} \, dt
\]

\[
= \int_{-\infty}^{\infty} h(w) e^{-j2\pi\mu(w + v)} \, dw \\
= e^{-j2\pi\mu v} \int_{-\infty}^{\infty} h(w) e^{-j2\pi\mu w} \, dw \\
= H(\mu) e^{-j2\pi\mu v}
\]

Let: \( w = t - v \)
For now...

- Thus:
  \[ \mathcal{F}\{h(t - v)\} = H(\mu)e^{-j2\pi\mu v} \]
  
- where \( H(\mu) \) is the Fourier transform of \( h(t) \) (the image)

- So:
  \[ \mathcal{F}\{f(t) \otimes h(t)\} = \int_{-\infty}^{\infty} f(v)[H(\mu)e^{-j2\pi\mu v}]dv \]
  
  \[ = H(\mu)\int_{-\infty}^{\infty} f(v)e^{-j2\pi\mu v}dv \]
  
  \[ = H(\mu)F(\mu) \]

- (Convolution of two functions in Spatial Domain) = (Product of two functions in Frequency Domain)
Convolution Theorem

- Fourier Transform Pair:

\[ f(t) \otimes h(t) \iff H(\mu)F(\mu) \]

\[ f(t)h(t) \iff H(\mu) \otimes F(\mu) \]

- Double-arrow means:

\[ \mathcal{Z}\{\text{left}\} = \text{right} \]

\[ \text{left} = \mathcal{Z}^{-1}\{\text{right}\} \]
SAMPLING AND QUANTIZATION
Sampling

- We have continuous function $f(t)$, and we want to sample at uniform intervals $\Delta T$.
Modeling Sampling as an Impulse Train

- We’ll use a sampling function $s(t)$ equal to an impulse train $\Delta T$ units apart
- Multiply $f(t)$ by $s(t)$ to get samples!

\[
\tilde{f}(t) = f(t) s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t) \delta(t - n\Delta T)
\]

Images from Gonzalez-Woods “Digital Image Processing”
Values of Individual Samples

- A specific sample $f_k$ is given by:

$$f_k = \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)dt$$

- because of the sifting property

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$$
The Story thus far...

- We want to get the Fourier transform of the sampled function

- We know $\tilde{f}(t)$ is the product of $f(t)$ and the impulse train $s(t)$
Bringing in Convolution Theorem

- Fourier transform of product of two functions in spatial domain = convolution of transforms of two functions in frequency domain
  \[ f(t)h(t) \Leftrightarrow H(\mu) \otimes F(\mu) \]
- ERGO:
  \[
  \tilde{F}(\mu) = \mathcal{Z}\{f(t)\} \\
  = \mathcal{Z}\{f(t)s_{\Delta T}(t)\} \\
  = F(\mu) \otimes S(\mu)
  \]
Bring back old friends...

- We know the Fourier transform of an impulse train:

\[
S(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)
\]

- We also know how convolution is defined:

\[
f(t) \otimes h(t) = \int_{-\infty}^{\infty} f(v)h(t-v)dv
\]
Fourier Transform of Sampled Function

\[ \tilde{F}(\mu) = F(\mu) \otimes S(\mu) \]

\[ = \int_{-\infty}^{\infty} F(v) S(\mu - v) dv \]

\[ = \frac{1}{\Delta T} \int_{-\infty}^{\infty} F(v) \sum_{n=-\infty}^{\infty} \delta \left( \mu - v - \frac{n}{\Delta T} \right) dv \]

\[ = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(v) \delta \left( \mu - v - \frac{n}{\Delta T} \right) dv \]

\[ = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F \left( \mu - \frac{n}{\Delta T} \right) \quad \text{Sifting property again} \]
So what does that mean?

- Fourier transform of sampled function =
  - *Infinite, periodic* sequence of *copies* of transform of original function
  - Value of $\frac{1}{\Delta T}$ determines separation between copies
  - *Note*: transform of sampled function is also continuous
DISCRETE FOURIER TRANSFORM (DFT) OF ONE VARIABLE
Continuous vs. Discrete

- Up to this point, Fourier transform of sampled function:
  - Infinite
  - Continuous

- In practice, have finite number of samples

- So, we’ll develop the **Discrete Fourier Transform (DFT)**
  - Again, sticking with 1 dimension for now
Deriving the Discrete Fourier Transform

- If we bring back the original definition of the Fourier transform and apply it to the sampled function, we get:

\[ \tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t} \, dt \]

- We’re going to replace the sampled function with the discrete version from before:

\[ \tilde{f}(t) = f(t) s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t) \delta(t - n\Delta T) \]
Deriving the DFT

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} \, dt$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t) \delta(t - n\Delta T) e^{-j2\pi\mu t} \, dt$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \delta(t - n\Delta T) e^{-j2\pi\mu t} \, dt$$

$$= \sum_{n=-\infty}^{\infty} f(n\Delta T) e^{-j2\pi\mu n\Delta T}$$

Sifting property

$$= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}$$
Characterizing Transform of Sampled Function

- $\tilde{F}(\mu)$ is continuous and infinitely periodic with period $1/\Delta T$

$$\tilde{F}(\mu) = \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi \mu n \Delta T}$$

- So, all we need to do is characterize $\tilde{F}(\mu)$ across one period
  - Sample across one period
Sampling Period of $\tilde{F}(\mu)$

- Let’s take $M$ equally spaced samples from $F(\mu)$ over period $\mu = 0$ to $\mu = 1/\Delta T$.

- Frequency values we want will be:

$$\mu = \frac{m}{M\Delta T} \quad m = 0, 1, 2, \ldots, M - 1$$
Discrete Fourier Transform

\[ F_m = \sum_{n=0}^{M-1} f_n e^{-j\frac{2\pi mn}{M}} \]
Inverse Discrete Fourier Transform

- Given our samples \{F_m\}, we can get the original samples \{f_n\} with the IDFT:

\[
f_n = \frac{1}{M} \sum_{m=0}^{M-1} F_m e^{j2\pi mn/M} \quad n = 0, 1, 2, ..., M - 1
\]
Discrete Fourier Transform Pair

- **Forward:**

\[
F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \quad u = 0, 1, 2, \ldots, M-1
\]

- **Inverse:**

\[
f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \quad x = 0, 1, 2, \ldots, M-1
\]

- **Note:** neither depend on \(\Delta T\), so this works on **any** finite, uniformly-sampled discrete data!
Reminder: Euler’s Formula

- You may be already thinking about how to implement this, and you might ask what to do with the $e^{j\cdots}$ part of the formula…

- That’s why we bring in Euler’s Formula!

\[ e^{j\theta} = \cos \theta + j \sin \theta \]

- This gives us a nice complex number to deal with:

\[ e^{-j2\pi ux/M} = \cos \left( \frac{-2\pi ux}{M} \right) + j \sin \left( \frac{-2\pi ux}{M} \right) \]