INTRODUCTION

• Points in space are grand, but how do we deal with lines, curves, and surfaces?

• In these slides, we’ll talk about some mathematical descriptions of these latter three things...
2D IMPLICIT CURVES
A lot of times we’ll see an equation (say, for a line) where you can directly get one of the coordinates (*dependent variable*) by plugging in the other (*independent variable*) $\rightarrow$ this would be the *explicit form*: 

$$y = x + 1$$

However, if you put all the variables (and constants) on the left side, such that the right side equals zero, you have the *implicit form*:

$$y - x - 1 = 0$$

$$f(x, y) = 0$$
IMPLICIT FORM OF A CURVE

• Curve
  • Set of points that can be drawn without lifting the pen/pencil from the page
  • 2D implicit form of a curve: \( f(x, y) = 0 \)
    • If the value of \( f(x, y) = 0 \) \( \rightarrow \) point \((x, y)\) ON the curve
    • Otherwise \( \rightarrow \) point \((x, y)\) NOT on the curve

• Example: circle with center at \((2, 3)\) with radius 4
  • If point \((x, y)\) on circle, \( f(x, y) \) will equal zero

\[
f(x, y) = (x - 2)^2 + (y - 3)^2 - 4^2
\]
WHY WOULD WE USE THIS FORM?

• One very useful property → depending on the value of \( f(x,y) \), we can determine whether \((x,y)\) is INSIDE or OUTSIDE the curve:

  • **Partitions space into:**
    • \( f = 0 \) → ON CURVE
    • \( f < 0 \) → “INSIDE” CURVE
    • \( f > 0 \) → “OUTSIDE” CURVE
VECTOR FORM OF THE IMPLICIT CIRCLE

• We can convert this into a form that uses vectors (instead of x and y coordinates directly):
  • \( P = \text{point we are evaluating} \)
  • \( C = \text{center of the circle} \)

• Noticing further that the dot product is the length squared:

\[
f(x, y) = (P - C) \cdot (P - C) - r^2 = 0
\]

• The radius is always positive, as is length, so:

\[
f(x, y) = \|P - C\|^2 - r^2 = 0
\]

• What does this mean? \(\Rightarrow\) A POINT is on circle IF the length of vector from the CENTER to the POINT is equal to the RADIUS
ADVANTAGES OF VECTOR FORM OF FORMULAS

• Vector form can be:
  • More intuitive
  • Allows you to extend the concept of formula to a different number of dimensions
    • E.g., previous formula
      • 2D → circle
      • 3D → sphere
  • Makes code cleaner and less error-prone
THE 2D GRADIENT
THE GRADIENT

• Let’s say we have a function $f$

• **Gradient of $f$**
  
  • Vector that points in the direction of **GREATEST CHANGE of $f$**
  
  • Each component $\rightarrow$ partial derivative of each dimension with respect to $f$
  
  • **Example**: 2D implicit function $f$
    
    • *First part of gradient*: change in $f$ when we mess with $x$ (holding $y$ constant)
    
    • *Second part of gradient*: change in $f$ when we mess with $y$ (holding $x$ constant)
GRADIENT = NORMAL!

- Vector is PERPENDICULAR/ORTHOGONAL to curve/surface at that point
  - Basically pointing in direction that would get you far away from the curve/surface the fastest

- Another name for the gradient is the NORMAL VECTOR
  - Perpendicular to another vector called the tangent vector, which points in the direction that keeps you ON the curve/surface

\[

\nabla f(x, y) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)

\]

2D Circle

\[
f(x, y) = (x - 2)^2 + (y - 3)^2 - 4^2
\]

Partial deriv. in x

\[
\frac{\partial f}{\partial x} = 2(x - 2) = 2x - 4
\]

Partial deriv. in y

\[
\frac{\partial f}{\partial y} = 2(y - 3) = 2y - 6
\]

Normal for 2D Circle centered at (2,3)

\[
\nabla f(x, y) = (2x - 4, 2y - 6)
\]

Normal at \((-2\sqrt{2} + 2, 2\sqrt{2} + 3)\)

\[
\nabla f(-1,5) = \left(2(-2\sqrt{2} + 2) - 4, 2(2\sqrt{2} + 3) - 6\right)
\]

\[
= (-4\sqrt{2} + 4 - 4, 4\sqrt{2} + 6 - 6)
\]

\[
= (-4\sqrt{2}, 4\sqrt{2})
\]
VECTOR FORM OF THE CIRCLE GRADIENT...

• IF WE ONLY CARE ABOUT THE DIRECTION OF THE NORMAL, we can divide the gradient of the 2D circle by some constant, like 2:

\[
\nabla f(x, y) = \frac{1}{2}(2x - 4, 2y - 6) = (x - 2, y - 3)
\]

• Basically, this is telling us that the direction of the circle gradient = vector from the CENTER of the circle to the POINT:

\[
\nabla f(x, y) = P - C
\]

• This can also be extended to higher dimensions, like spheres 😊
IMPLICIT 2D LINES
POINT OF CLARITY: LINES VS. VECTORS

• Before we go on, to be clear:
  • **Vectors**
    • Have finite length
    • Do not (necessarily) go through a specific point in space
  • **Lines**
    • Have INFINITE length
    • DO go through specific points in space

• We will use vectors in some of our equations for lines, but I wanted to clarify the difference between them.
You’ve all seen the familiar slope-intercept form of a 2D line:

- **b** = y-intercept → where line cross Y axis
- **m** = slope → “rise” over “run”
  - If **a** = where the line cross the X axis, then: \( m = -\frac{b}{a} \)

One disadvantage of this form: could have infinite slope (vertical line)
SLOPE-INTERCEPT $\Rightarrow$ IMPLICIT FORM

• This can be converted to the implicit form: $y - mx - b = 0$

• Notion of:
  • “Outside” $\Rightarrow f(x,y) > 0 \Rightarrow “Above”$
  • “Inside” $\Rightarrow f(x,y) < 0 \Rightarrow “Below”$

• However, you will notice we are still using the slope to construct it...
GENERAL (IMPLICIT) FORM

• To avoid the slope issues, the general form is more useful:

\[ Ax + By + C = 0 \]

• A,B,C → real number constants
  • QUICK ASIDE: We can multiply this formula by any constant (except 0) and get the same line

• BUT...how do we find the formula for a line in this form, given two points on the line?
Let’s say we have two points: \((x_0, y_0)\) \((x_1, y_1)\)

Plugging these points in gives us two equations:

\[Ax_0 + By_0 + C = 0\]
\[Ax_1 + By_1 + C = 0\]

BUT we have THREE unknowns!
- A, B, and C

Simply put...now what?
GENERAL FORM AND THE GRADIENT

• Turns out, the general form has a VERY nice property $\rightarrow$ (A,B) is the GRADIENT VECTOR!

• We know the gradient is PERPENDICULAR to the line

• A vector that is PARALLEL with the line can be formed from the two points: $(x_1 - x_0, y_1 - y_0)$

• Given a vector $(x,y)$, a vector perpendicular to it can be: $(y,-x)$ OR $(-y,x)$

• SO, we’ll say the gradient is: $(y_0 - y_1, x_1 - x_0) = (A,B)$
  • Which gives us A and B!!!
SOLVING FOR THE GENERAL FORM
(FOR REAL THIS TIME)

• Plugging in A and B:

\[(y_0 - y_1)x + (x_1 - x_0)y + C = 0\]

• Now we can solve for C by plugging in either of our points

• If we use the first one:

\[
\begin{align*}
C &= -(y_0 - y_1)x_0 - (x_1 - x_0)y_0 \\
C &= -y_0x_0 + y_1x_0 - x_1y_0 + x_0y_0 \\
C &= x_0y_1 - x_1y_0
\end{align*}
\]
FINAL FORM OF THE GENERAL FORM (GIVEN TWO POINTS)

\[(y_0 - y_1)x + (x_1 - x_0)y + x_0y_1 - x_1y_0 = 0\]

• One nice property: no division \(\rightarrow\) no possible division by zero

• NOTE: This is just ONE of INFINITELY many possible representations of this line
  • ...but this will do.
You can also convert the general form back to slope-intercept form (if it exists):

- Move non-y terms to right-hand side
- Divide by multiplier of y term

\[
(y_0 - y_1)x + (x_1 - x_0)y + x_0y_1 - x_1y_0 = 0
\]

\[
(x_1 - x_0)y = -(y_0 - y_1)x - x_0y_1 + x_1y_0
\]

\[
y = \frac{-(y_0 - y_1)}{(x_1 - x_0)} x + \frac{x_1y_0 - x_0y_1}{(x_1 - x_0)}
\]

\[
y = \frac{(y_1 - y_0)}{(x_1 - x_0)} x + \frac{x_1y_0 - x_0y_1}{(x_1 - x_0)}
\]
**DISTANCE FROM THE LINE**

- The general implicit form can also be used to compute the SIGNED distance of a point from the line
  - \( f(x, y) \) is PROPORTIONAL to that distance \( \rightarrow \) we don’t know what the scales of \( A \) and \( B \) are, so we can’t use \( f(x, y) \) by itself
- The distance from a point \( P \) to the line \( \rightarrow \) length of the normal vector times some constant \( k \):

\[
\text{distance} = k\sqrt{A^2 + B^2}
\]

- We can think of our point \( P \) as being some point \( Q \) on the line + a scaled version of the normal:

\[
P = Q + k(A, B)
\]
DISTANCE FROM THE LINE

• Plugging P into the line equation:

\[ f(p_x, p_y) = f(q_x + kA, q_y + kB) \]
\[ = A(q_x + kA) + B(q_y + kB) + C \]
\[ = Aq_x + kA^2 + Bq_y + kB^2 + C \]
\[ = (Aq_x + Bq_y + C) + kA^2 + kB^2 \]
\[ = kA^2 + kB^2 \]
\[ = k(A^2 + B^2) \]

NOTE: Q is on the line, so the part in parentheses is equal to zero \( \rightarrow f(Q) = 0 \)
DISTANCE FROM THE LINE

• So now we know:

\[ f(p_x, p_y) = k(A^2 + B^2) \]

• And we already know:

\[ \text{distance} = k\sqrt{A^2 + B^2} \]

• So, if we take the latter formula and do a little finagling:

\[
\text{distance} = k\sqrt{A^2 + B^2} = k\sqrt{A^2 + B^2} \frac{\sqrt{A^2 + B^2}}{\sqrt{A^2 + B^2}} = \frac{k(A^2 + B^2)}{\sqrt{A^2 + B^2}} = \frac{f(p_x, p_y)}{\sqrt{A^2 + B^2}}
\]
DISTANCE FROM LINE FORMULA: FINAL FORM

\[ \text{distance} = \frac{f(p_x, p_y)}{\sqrt{A^2 + B^2}} \]

- This gives you the ACTUAL, GEOMETRIC, SIGNED distance of a point \( P \) from the line.
- IF you normalize \((A, B)\) (which would entail dividing ALL of \( f(x, y) \) by the length of \((A, B)\)), THEN:

\[ \text{distance if (A, B) normalized} = f(p_x, p_y) \]
IMPLICIT QUADRIC CURVES
QUADRIC CURVES

• If we have a second-order term in our curve, we are dealing with a **Quadric Curve** (or **Quadrics**):
  • Circles
  • Ellipses
  • Hyperbolas
  • Parabolas
• All these would be considered **conic sections**
CONIC SECTIONS

• A conic section (or conic) can be described with this second-degree implicit form equation:

\[ Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \]

• Basically, it’s any (nondegenerate) curve generated by the intersection of a plane with a top and/or bottom of a double cone
• “Degenerate” conics would be things like points and straight lines
• The values of A, B, C, D, E, and F determine what kind of curve we want
• If we look at \((C^2 - 4AB)\):

\[
C^2 - 4AB \begin{cases} 
< 0, & \text{ellipse (or circle)} \\
= 0, & \text{parabola} \\
> 0, & \text{hyperbola}
\end{cases}
\]
As we’ve seen, a circle can be defined by its implicit form:

- \((x_c, y_c) \rightarrow \text{center of circle}\)
- \(r = \text{radius}\)

\[
f(x, y) = (x - x_c)^2 + (y - y_c)^2 - r^2 = 0
\]
ELLIPSES

• Ellipses are basically elongated circles

• **Formal definition:** given two points \( F_1(x_1, y_1) \) and \( F_2(x_2, y_2) \) (the *foci* of the ellipses), a point \((x,y)\) is on the ellipse if the sum of the distances from \( F_1 \) and \( F_2 \) is some constant \( c \).

\[
d_1 + d_2 = c
\]

• Conceptually, this is like having a string whose ends are connected at \( F_1 \) and \( F_2 \), while you use a pencil to trace out the ellipse.

• If we expand this out, we get the *implicit form of an ellipse*:

\[
f(x, y) = \sqrt{(x-x_1)^2 + (y-y_1)^2} + \sqrt{(x-x_2)^2 + (y-y_2)^2} - c
\]
AXIS-ALIGNED ELLIPSES

- Ellipses have two axes:
  - **Major axis** = line passing through both foci
  - **Minor axis** = line perpendicular to major axis
- The **semimajor** and **seminor** axes are like radii aligned with the major and minor axes, respectively
- If the ellipse axes are aligned with the coordinate axes (i.e., x and y), the equation we can use is a lot simpler:

\[
f(x, y) = \left( \frac{x - x_c}{r_x} \right)^2 + \left( \frac{y - y_c}{r_y} \right)^2 - 1 = 0
\]

- ...where
  - \( r_x = \) semimajor/minor axis aligned with x axis
  - \( r_y = \) semimajor/minor axis aligned with y axis
3D IMPLICIT SURFACES AND THE NORMAL
• Just as we can define 2D curves implicitly → we can also define 3D surfaces!

\[ f(x, y, z) = 0 \]

• If point \((x, y, z)\) is:
  • ON the surface → \(f(x, y, z) = 0\)
  • UNDER/INSIDE the surface → \(f(x, y, z) < 0\)
  • OVER/OUTSIDE the surface → \(f(x, y, z) > 0\)

• NOTE: Cannot always explicitly construct points from the implicit formula
  • I.e., cannot generate points that exist on the surface
IMPLICIT VECTOR FORM

• The vector form of the implicit formula is:

$$f(P) = 0$$

• Where $P = (x, y, z)$
  • ...although this could also be 2D if $P = (x, y)$
SURFACE NORMAL

- **Surface normal**
  - Vector perpendicular to the surface
  - Same as the 3D gradient
  - **NOTE:** The surface normal may change depending on where you are on the surface!!!
    - E.g., a sphere (changes) vs. a plane (doesn’t change)
    - (SHOULD) point towards the OUTSIDE of the surface
      - If it doesn’t, flip the sign of the function \( f \)

\[
N = \nabla f (P) = \left( \frac{\partial f (P)}{\partial x}, \frac{\partial f (P)}{\partial y}, \frac{\partial f (P)}{\partial z} \right)
\]

Gradient of 3D Implicit Function
IMPLICIT PLANES
3D PLANES

- The most basic 3D surface we can make is a **3D plane**
  - Extends infinitely along the plane
  - Divides all space into “inside” and “outside”

- A plane may be defined by:
  - A single point \( A \) on the plane \( f(P) = 0 \)
  - The surface normal \( N \)

- **NOTE**: We know the values of \( A \) and \( N \); 
  \( P \) is the unknown point we are testing...

\[
f(P) = (P - A) \cdot N = 0
\]

- The formula for a plane written this way is called the **point-normal form**
POINT-NORMAL FORM EXPLAINED

• Let’s take a closer look at what’s happening with the point-normal form...
  • We’re subtracting a point on the plane $A$ from $P$ → gives us a vector going from $A$ to $P$
    • Another way of thinking about it: $A$ is our new “origin” → getting $P$’s coordinates relative to the new origin $A$
  • Then, we’re getting the dot product of $(P - A)$ with the normal $N$ → orthogonally projecting $(P - A)$ onto $N$
    • Gives us the distance (times the length of $N$) that $(P - A)$ goes above or below the plane
    • If $(P - A)$ dotted with $N = 0$ → $(P - A)$ and $N$ are perpendicular → $P$ is on the plane!
DEFINING A PLANE WITH 3 POINTS

• Suppose instead of a point and normal, we have 3 points A, B, C on the plane
• Any vector from any pair of A, B, or C will be a vector on the plane
• So, we can get the normal \( \Rightarrow \) cross product of \( (B - A) \) and \( (C - A) \):

\[
N = (B - A) \times (C - A)
\]

• So, our implicit plane equation becomes

\[
f(P) = (P - A) \cdot ((B - A) \times (C - A)) = 0
\]
ASIDE: SCALAR TRIPLE PRODUCT

• The **scalar triple product** of three vectors $a$, $b$, and $c$ is given by:

$$a \cdot (b \times c) = 0$$

• Turns out the **absolute value of the scalar triple product** = **volume of the parallelepiped** formed by those three vectors:

$$|a \cdot (b \times c)| = \text{volume of parallelepiped}$$

• Notice that, in our previous plane equation, if $P$ is on the plane $\rightarrow$ all vectors are on the plane $\rightarrow$ scalar triple product equals zero $\rightarrow$ volume of parallelepiped is ZERO:

$$f(P) = (P - A) \cdot ((B - A) \times (C - A)) = 0$$

http://farside.ph.utexas.edu/teaching/em/lectures/img172.png
The general implicit form for a 3D plane is given by:

$$Ax + By + Cz + D = 0$$

- Note how similar it is to the general implicit form for a 2D line.

- ALSO (like the 2D line equation) $$(A,B,C) = \text{GRADIENT} = \text{NORMAL}$$
SOLVING THE GENERAL IMPLICIT PLANE EQUATION

• If we already know the normal:
  • Just plug in (A,B,C) and solve for D

• If we DON’T know the normal, BUT we do know 3 non-collinear points on the plane \( (P_1, P_2, P_3) \):
  • EITHER
    • Compute normal from cross product of \((P_2 - P_1) \) and \((P_3 - P_1) \)
    • Flip normal if necessary to get the desired inside/outside direction
    • Plug in normal \((A,B,C)\) and solve for D
  • OR
    • Divide equation by D
    • Solve for 3 unknowns \((A/D, B/D, \text{ and } C/D)\) with 3 equations (you can use Cramer’s Rule for this…)

\[
\frac{(A/D)x_k + (B/D)y_k + (C/D)z_k}{k = 1,2,3} = -1
\]
ON/BEHIND/IN FRONT OF PLANE

• No matter which implicit form we use:
  • \( f(x,y,z) = 0 \) \( \rightarrow \) ON PLANE
  • \( f(x,y,z) < 0 \) \( \rightarrow \) BEHIND PLANE
  • \( f(x,y,z) > 0 \) \( \rightarrow \) IN FRONT OF PLANE

(1,0,1) is on the plane:
\[
(1,0,0) \cdot [(1,0,1) - (1,1,1)]
= (1,0,0) \cdot (0,-1,0)
= 0
\]

(0,0,1) is NOT on the plane:
\[
(1,0,0) \cdot [(0,0,1) - (1,1,1)]
= (1,0,0) \cdot (-1,-1,0)
= -1 \neq 0
\]
OTHER 3D IMPLICIT SURFACES
3D QUADRIC SURFACES

Not surprisingly, you can define 3D quadric surfaces implicitly:

- **3D sphere**:
  \[ f(P) = (P - C)^2 - r^2 = 0 \]

- **3D axis-aligned ellipsoid**:
  \[ f(P) = \frac{(x - x_c)^2}{a^2} + \frac{(y - y_c)^2}{b^2} + \frac{(z - z_c)^2}{c^2} - 1 = 0 \]
3D CURVES FROM IMPLICIT SURFACES

• Can we use 3D implicit formula $f(P) = 0$ to generate 3D curves (like, lines floating in space)?
  • No $\rightarrow$ really becomes a degenerate surface

• Example: use intersection of two planes to form a 3D line (both equations true $\rightarrow$ on line)

  \[
  f(P) = 0 \\
  g(P) = 0
  \]

• Better idea: use parametric curves...

2D PARAMETRIC CURVES
PARAMETRIC CURVES

• **Parametric curve**
  - Curve controlled by a single parameter $t$ tells you how far along the curve you are
  - Each dimension (x, y, and z) now become different functions of $t$:
    $x = g(t)$
    $y = h(t)$
  - Or, in vector form: $P = f(t)$
  - As long as the functions (like $g()$ and $h()$) are continuous, get continuous curve
  - Also, can use values of $t$ to generate points on curve!
PARAMETRIC CURVE AS MOVING POINT

- Parametric curves can go anywhere!
  - E.g., loop back over/under itself, crossing itself, etc.

- One can think of a parametric curve as:
  - \( t = \text{TIME} \)
  - \( f(t) = \text{function of time} = \text{point location at particular time} \)
  - Point is “moving” (and could move slower or faster at any given time)

Example from: http://www.math.harvard.edu/archive/21a_summer_03/labs/lab.html/
2D PARAMETRIC LINES: 2 POINTS ON LINE

Given a 2D line that passes through two points:

\[ \begin{align*}
  P_0 &= (x_0, y_0) \\
  P_1 &= (x_1, y_1)
\end{align*} \]

This can be written as:

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} = \begin{bmatrix}
  x_0 + t(x_1 - x_0) \\
  y_0 + t(y_1 - y_0)
\end{bmatrix}
\]

OR in vector form:

\[ P(t) = P_0 + t(P_1 - P_0) \]
2D PARAMETRIC LINES: A CLOSER LOOK

• First of all, notice what happens with certain key values of $t$:
  • $t = 0$ \(\rightarrow\) BEGINNING of line segment \((P_0)\)
  • $t = 1$ \(\rightarrow\) END of line segment \((P_1)\)
  • $t = 0.5$ \(\rightarrow\) half-way between \(P_0\) and \(P_1\)

• Also note:
  • $0 \leq t \leq 1$ \(\rightarrow\) ON or BETWEEN \(P_0\) and \(P_1\)
  • $t < 0$ \(\rightarrow\) “far side” of \(P_0\)
  • $t > 1$ \(\rightarrow\) “far side” of \(P_1\)

• “Start at \(P_0\), and go some distance towards \(P_1\) determined by parameter \(t\)”
  • $t = \frac{\text{fractional distance between points}}{...}$ although not necessarily ACTUAL distance

\[
P(0) = P_0 + 0(P_1 - P_0) = P_0
\]

\[
P(1) = P_0 + (P_1 - P_0) = P_0 - P_0 + P_1 = P_1
\]
Another way to look at 2D parametric lines: starting point $Q$ and vector $D$:

$$P(t) = Q + tD$$

- If $D$ = unit vector $\rightarrow$ line is **arc-length parameterized**
  - Means $t$ = EXACT measure of distance along line
  - In this case, “arc length” = actual length of curve itself
2D PARAMETRIC CIRCLES AND ELLIPSES

• For both circles and ellipses, we will use $\phi$ as our parameter
  • $\phi = \text{angle around x axis}$
  • To ensure points are unique, restrict $\phi$ to any interval of length $2\pi$:
    $$\phi \in [0, 2\pi)$$
  • Should look familiar $\Rightarrow$ effectively **polar coordinates** with fixed radius/radii

• **Circles**
  • To make a 2D parametric circle with center $(x_c, y_c)$ and radius $r$:
    $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_c + r \cos \phi \\ y_c + r \sin \phi \end{bmatrix}$

• **Axis-aligned Ellipse**
  • Two different “radii” (semimajor and semiminor axes lengths):
    $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_c + a \cos \phi \\ y_c + b \sin \phi \end{bmatrix}$
3D PARAMETRIC CURVES
3D PARAMETRIC CURVES

- The 3D parametric curves are very similar to 2D:

\[
\begin{align*}
x &= f(t) \\
y &= g(t) \\
z &= h(t)
\end{align*}
\]

- Domain: \( R \)
- Range: \( R^3 \)
- Example: 3D spiral around z-axis:

\[
\begin{align*}
x &= \cos t \\
y &= \sin t \\
z &= t
\end{align*}
\]

Visualized with K3DSurf: http://k3dsurf.sourceforge.net/
3D PARAMETRIC LINES

• We can write a 3D parametric line in vector form in the same two ways we did for 2D parametric lines:
  
  • Interpolation between two 3D points $P_0$ and $P_1$:
    • Often used to define a LINE SEGMENT
    
    $$P(t) = P_0 + t(P_1 - P_0)$$
  
  • Starting 3D point $Q$ and 3D direction vector $D$:
    • Often used to define a RAY
      • Ray = starts at point, goes off infinitely in one direction
    
    $$P(t) = Q + tD$$
MORE ADVANCED 3D PARAMETRIC CURVES

• In the future, we will talk more about advanced 3D parametric curves (splines, Bezier curves, etc.)
  • For now, though, we’ll stick with 3D lines and rays...
3D PARAMETRIC SURFACES
3D PARAMETRIC SURFACES

• We can define a surface in 3D parametrically; HOWEVER, we now need TWO parameters \( u \) and \( v \)
  • Before with curves \( \rightarrow \) only worried about how far along we were on the curve \( \rightarrow \) can describe with one parameter
  • With surfaces \( \rightarrow \) can travel in TWO different orthogonal directions \( \rightarrow \) need two parameters to describe that
• Thus, our general form is:
  \[
  \begin{align*}
  x &= f(u, v) \\
  y &= g(u, v) \\
  z &= h(u, v)
  \end{align*}
  \]

• Or, in vector form:
  \[
  [x, y, z] = P(u, v)
  \]

\[ P : R^2 \rightarrow R^3 \]
WHY WE NEED TWO PARAMETERS

• **Curves**  →  just need one parameter

• **Surfaces**  →  need two parameters
We can describe a sphere parametrically:

- Think of the two parameters like latitude and longitude on Earth
- \((r \sin \theta)\) \(\rightarrow\) effectively “radius” in the XY plane
- If \(r\) were not fixed \(\rightarrow\) \((r, \theta, \phi) = \text{spherical coordinates}\)

To get \((\theta, \phi)\) from \((x,y,z)\):

\[
\theta = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)
\]
\[
\phi = \arctan2(y, x)
\]

\[
x = (r \sin \theta) \cos \varphi
\]
\[
y = (r \sin \theta) \sin \varphi
\]
\[
z = r \cos \theta
\]
DERIVATIVES OF P

• Before we took the derivative of f \( \rightarrow \) got the gradient

• What happens when we take the derivative of \( P(u,v) \)?
  • Let’s work through this...
  • Derivative of parametric CURVE \( Q(t) \) \( \rightarrow \) vector TANGENT to curve
  • If we lock the \( v \) parameter (set it to \( v_0 \)), then \( P(u, v_0) = Q(u) = \text{curve inside of surface (going in u direction)} \)
    • This kind of curve with constant \( u \) or \( v \) value = isoparametric curve (or isoparm)
  • Take derivative of \( Q(u) \) \( \rightarrow \) partial derivative of \( P(u,v) \) with respect to \( u \) \( \rightarrow P_u \)
  • Do the same thing with respect to \( v \) \( \rightarrow P_v \)
  • We now have two TANGENT vectors on the surface that point along \( u \) and \( v \), respectively: \( P_u \) and \( P_v \)

• SO, to get the normal, just take the cross product!

\[ N = P_u \times P_v \]
SUMMARY OF CURVES AND SURFACES
SUMMARY OF FORMS

• Implicit curves in 2D:
  \[ f : \mathbb{R}^2 \mapsto \mathbb{R} \]
  \[ S = \{ P \mid f(P) = 0 \} \]

• Implicit surfaces in 3D:
  \[ f : \mathbb{R}^3 \mapsto \mathbb{R} \]

• Parametric curve in 2D:
  \[ P : D \subset \mathbb{R} \mapsto \mathbb{R}^2 \]
  \[ S = \{ P(t) \mid t \in D \} \]

• Parametric curve in 3D:
  \[ P : D \subset \mathbb{R} \mapsto \mathbb{R}^3 \]

• Parametric surface in 3D:
  \[ P : D \subset \mathbb{R}^2 \mapsto \mathbb{R}^3 \]
  \[ S = \{ P(u,v) \mid (u,v) \in D \} \]
SUMMARY OF DERIVATIVES

• *Implicit curves/surfaces*
  • Derivative of $f \rightarrow$ GRADIENT = NORMAL
  • Tangent can be calculated from basis formed by normal

• *Parametric curves/surfaces*
  • Derivative of $P \rightarrow$ TANGENT vector(s)
  • Normal can be calculated from basis formed by tangent(s)
LINEAR INTERPOLATION
LINEAR INTERPOLATION

• **Linear interpolation**
  • Possibly the most common mathematical operation in graphics
  • Basically means:
    • Have parameter $t$ that varies from 0 to 1
    • $t = 0 \rightarrow$ point A
    • $t = 1 \rightarrow$ point B
    • $0 < t < 1 \rightarrow$ some point between A and B given by: $(1 - t)A + tB$
  • *Linear* in $t$ → only first-order terms in $t$
  • *Interpolation* → goes through points A and B exactly (at $t = 0$ and $t = 1$, respectively)
  • We were using linear interpolation for parametric lines before
ANOTHER USE OF LINEAR INTERPOLATION

• Have a set of data points \((x_i, y_i)\)

• Want to create a continuous function that connect all these points in a “curve”

• Basically use linear interpolation to create lines between each pair of points \((x_i, y_i)\) and \((x_{i+1}, y_{i+1})\)

\[
f_i(x) = y_i + \frac{x - x_i}{x_{i+1} - x_i}(y_{i+1} - y_i)
\]

• The “t” here is basically how much of the x distance we’ve covered:

\[
t = \frac{x - x_i}{x_{i+1} - x_i}
\]