Chaos in Dynamical Systems
Overview

• Introduction: Modeling Nature!
• Example: Logistic Growth
• Fixed Points
• Bifurcation Diagrams
• Application Examples
INTRODUCTION
Linear and Non-linear dynamic systems

• Behavior evolves with time
• Dynamic analysis / time series analysis
  – trends, forecasts
• Causality models
  – Cause $\rightarrow$ Effect (not immediately!)
• Time:
  – Continuous $\rightarrow$ differential equations
  – Discrete $\rightarrow$ difference equations
Example: Biological Modelling

- Population of bacteria growth, doubles every hour:
  \[ x_n = f(x_{n-1}) = 2x_n \]

- A deterministic, dynamical system \((n: \text{discrete time})\)

- Question: What happens to, say, \(f^2(x) = f(f(x))\) ?

- In general: \(f^k(x)\) ?

- Easy … and unrealistic!
Example: A Better Model

• Available space/resources are limited:

\[ g(x) = 2x(1-x) \]

• Small \( x \): Exponential growth as before

• Large \( x \): Nonlinear effect involving “remaining space” \((1-x)\)

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EXAMPLE: LOGISTIC GROWTH
Excursion: Logistic Growth

Time continuous → differential equation

\[ y' = \frac{dy}{dt} = c \cdot y \cdot (1 - \frac{y}{a}) \]

By integration:

\[ y(t) = \frac{a}{1 + e^{-c \cdot (t - T)}} \]

Limits:

\[ \lim_{t \to \infty} y(t) = a \quad \lim_{t \to -\infty} y(t) = 0 \]
Logistic Growth Examples

- Ecology: Population growth into a niche
- Economy: Market penetration by some goods
- Society: Learning curve
The discovery of the stable elements

\[ F = \frac{y}{a} \]
Car population in Italy

Car population in Italy (in millions)

\[ \alpha \]

\[ 0 - 1990 \]

\[ 1950 - 1970 \]

LIACS Natural Computing Group Leiden University
Car registration - Italy
British railways-displacement of coal by oil
Car production - Italy

Data Source: World MVMF Ass. Data, 1983

C. Marchetti, IIASA, 1984
USA – Substitution of horses by automobiles
World primary energy substitution
The mining industry in the US

- Very regular logistic progression can develop instabilities when the saturation point is approached
FIXED POINTS
Definition

- **Map**: Function $f$ whose domain and range are the same.
- The set of points $\{x, f(x), f^2(x),\ldots\}$ is called **orbit** of $x$ under $f$.
- **Initial value**: Starting point $x$ of the orbit.
- Point $p$ is called **fixed point** of map $f$, iff: $f(p) = p$
Cobweb Plots

- Used for mapping out an orbit
Cobweb Plots

- Orbits beginning near fixed points act differently.

\[ f(x) = \frac{3x - x^3}{2} \]
Classes of Fixed Points

• Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $f(p) = p$.

• **Sink, attracting fixed point**: If all points sufficiently close to $p$ are attracted by $p$:

\[
\exists \varepsilon > 0 : \forall x \in N_\varepsilon(p) : \lim_{k \to \infty} f^k(x) = p
\]

• **Source, repelling fixed point**:

\[
\exists \varepsilon > 0 : \forall x \in N_\varepsilon(p), x \neq p : \lim_{k \to \infty} f^k(x) \notin N_\varepsilon(p)
\]
Theorem on Fixed Points

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ smooth, and $f(p) = p$.
- If $|f'(p)| < 1$, then $p$ is a sink.
- If $|f'(p)| > 1$, then $p$ is a source.

Proof: part 1. Let $a$ be any number between $|f'(p)|$ and 1. Since

$$\lim_{x \to p} \frac{|f(x) - f(p)|}{|x - p|} = |f'(p)|$$

there is a neighborhood $N_\varepsilon(p)$ such that

$$\frac{|f(x) - f(p)|}{|x - p|} < a$$

For $x \in N_\varepsilon(p)$. This means, $f(x)$ is closer to $p$ than $x$ is, by at least a factor of $a$. Also,

$$|f^k(x) - p| \leq a^k|x - p|$$

i.e., $p$ is a sink.
Periodic Points

• Let’s start changing $a$, say, $a=3.3$.
• Fixed points:
  – $X=0$
  – $X=23/33=0.696969…$
  – Both are repellers!
  – Where do the orbits go?
Definition: Periodic Point

- $p$ is called a **periodic point of period** $k$, if $f^k(p) = p$ and if $k$ is the smallest such positive integer.
- The orbit with initial point $p$ is called a **periodic orbit** of period $k$.
- The orbit consists of $k$ points.
- The period-$k$ orbit of $p$ is a
  - periodic sink if $p$ is a sink for the map $f^k$.
  - Periodic source if $p$ is a source for the map $f^k$. 
Stability Test for Periodic Orbits

• The periodic orbit \( \{ p_1, \ldots, p_k \} \) is a sink, if

\[
|f'(p_k) \cdots f'(p_1)| < 1
\]

• and a source, if

\[
|f'(p_k) \cdots f'(p_1)| > 1
\]

This is because

\[
(f^k)'(p_1) = (f(f^{k-1}))'(p_1)
\]
\[
= f'(f^{k-1}(p_1))(f^{k-1})'(p_1)
\]
\[
= f'(f^{k-1}(p_1))f'(f^{k-2}(p_1)) \cdots f'(p_1)
\]
\[
= f'(p_k) \cdots f'(p_1)
\]
BIFURCATION DIAGRAMS
Bifurcation Diagram $g_a(x) = ax(1-x)$

- Computer algorithm:
  1. Choose a value of $a$, starting with $a=1$.
  2. Choose $x$ at random in $[0,1]$.
  3. Calculate the orbit of $x$ under $g_a(x)$.
  4. Ignore the first 100 iterations and plot the orbit, beginning with iterate 101.
  5. Increment $a$, go to step 2.
Bifurcation Diagram $g_a(x) = ax(1-x)$
\[ g_4(x) = 4x(1-x) \]

- Now it gets really interesting …
- For each positive integer \( k \), there is an orbit of period \( k \) … infinitely many …
- No idea, where we end up – based on starting point.
- Two initial conditions close together can get us to orbits which are far apart from each other
- See Excel simulation …
- This is what we mean by “deterministic chaos”
Bifurcation Cascades

• If the $n$th period-doubling occurs at $a = a_n$:

\[ \lim_{n \to \infty} \frac{a_{n-1} - a_{n-2}}{a_n - a_{n-1}} = 4.669201609... \]

• This is the same for any one-parameter family of unimodal maps with negative so-called Schwarzian derivative.

• Feigenbaum’s constant

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APPLICATION EXAMPLE
Application Example

• Population dynamics of flour beetles (Tribolium)
• Newly hatched larva spends 2wks feeding before entering a pupa stage of same length
• Beetle exits pupa stage as an adult
• Model:
  - $L_t$: number of larvae at time $t$.
  - $P_t$: number of pupae at time $t$.
  - $A_t$: number of adults at time $t$.
  - Time unit is 2wks
The Model

- \( b \): Birth rate
- \( \mu_l, \mu_p, \mu_a \): Death rates (larva, pupa, adult)

\[
L_{t+1} = bA_t \\
P_{t+1} = L_t(1 - \mu_l) \\
A_{t+1} = P_t(1 - \mu_p) + A_t(1 - \mu_a)
\]
Extended Model

• But … also cannibalism if overpopulated
  - Adults eat pupae and unhatched eggs (future larvae)
  - Larvae also eats eggs (future larvae)

\[
L_{t+1} = bA_t \exp\left(-c_{ea} A_t - c_{el} L_t\right)
\]
\[
P_{t+1} = L_t \left(1 - \mu_l\right)
\]
\[
A_{t+1} = P_t \left(1 - \mu_p\right) \exp(-c_{pa} A_t) + A_t \left(1 - \mu_a\right)
\]
Parameters

- Found through population experiments:

\[
\begin{align*}
  c_{el} &= 0.012 \\
  c_{ea} &= 0.09 \\
  c_{pa} &= 0.004 \\
  \mu_l &= 0.267 \\
  \mu_p &= 0 \\
  b &= 7.48 \\
  \mu_a &= 0.0036
\end{align*}
\]
Dynamics - Experimental

- 4 experimental runs
- 18 consecutive 2wk periods
- 5 different adult mortality rates (by removing adults after 12wks)
- Behavior corresponds with model
Some more examples …

Period-doubling cascade from a CO$_2$-laser
Some more examples ...

Optical fiber laser
Some more examples...

Chua’s circuit
Conclusions

- Simple maps show very rich dynamical behaviour
- Can enter into fixed points / orbits
- 2 similar initial conditions can yield completely different behaviour
- Infinitely many orbits possible
- Still this is deterministic chaos!
- Many applications in natural system modeling
- E.g., compare fractals, cellular automata, …