General Dynamical Systems.

Obtain an equation of motion in the form

\[ \dot{x} = \frac{dx}{dt} = f(x, p, t) \]

and solve it over a parameter set \( p \).

Growth-decay models.

Let \( P(t) = \) population density at time \( t \).
Initial condition: at \( t = 0, p = p_0 \)

1. Constant decay: \( \frac{dp}{dt} = -k \)
   
   Solution: \( p = p_0 - kt \)

\[ p_0 \]

\[ t \]

2a. Exponential decay: \( \frac{dp}{dt} = -kp \)

   Solution: \( p = p_0 e^{-kt} \)
2b. Exponential growth: \[ \frac{dp}{dt} = kp, \quad p(t = 0) = p_0 \]

\[ p = p_0 e^{kt} \]

3. Movement towards a stable plateau

\[ \frac{dp}{dt} = a - kp \]

At equilibrium, rate is zero. Therefore, at equilibrium,

\[ p = p_\infty = \frac{a}{k} \]

Solution comes out as

\[ p = p_\infty - (p_\infty - p_0) e^{-kt} \]
Numerical techniques to solve differential equation models.

Euler method. A linear scheme to keep things simple.

Given is an equation like $\frac{dy}{dt} = f(y, t)$ for the slope of a curve we have yet to determine.

Using Taylor’s series form of a function, we note that the solution $y(t + h)$ appears as

$$y(t + h) = y(t) + h \left( \frac{dy}{dt} \right)_t + \frac{h^2}{2!} \left( \frac{d^2y}{dt^2} \right) + \ldots$$

Thus, at $t_{n+1}$, approximately,

$$y(t_{n+1}) = y_{n+1} = y_n + hf(y_n, t_n)$$
$$y_{n+2} = y_{n+1} + hf(y_{n+1}, t_{n+1})$$

And so on!

Solve the following using Euler method:
\[ y' = y - t^2 + 1 \] with the initial condition \( y_0 = 0.5 \). The exact solution is \( y(t) = (t + 1)^2 - 0.5e^t \).

Let \( h = 0.2 \), \( t_k = 0.2k \). We have \( f(y, t) = y - t^2 + 1 \)

With \( y_0 = 0.5 \), and \( y_{k+1} = y_k + h(y_k - t_k^2 + 1) \)

\[ = y_k + 0.2(y_k - 0.04k^2 + 1) \]

The solution profile:

<table>
<thead>
<tr>
<th>t</th>
<th>y(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.500000</td>
</tr>
<tr>
<td>1</td>
<td>0.800000</td>
</tr>
<tr>
<td>2</td>
<td>1.152000</td>
</tr>
<tr>
<td>3</td>
<td>1.550400</td>
</tr>
<tr>
<td>4</td>
<td>1.988480</td>
</tr>
<tr>
<td>5</td>
<td>2.458176</td>
</tr>
<tr>
<td>6</td>
<td>2.949811</td>
</tr>
<tr>
<td>7</td>
<td>3.451774</td>
</tr>
<tr>
<td>8</td>
<td>3.950128</td>
</tr>
<tr>
<td>9</td>
<td>4.428154</td>
</tr>
<tr>
<td>10</td>
<td>4.865785</td>
</tr>
<tr>
<td>11</td>
<td>5.238942</td>
</tr>
<tr>
<td>12</td>
<td>5.518730</td>
</tr>
<tr>
<td>13</td>
<td>5.670476</td>
</tr>
<tr>
<td>14</td>
<td>5.652571</td>
</tr>
<tr>
<td>15</td>
<td>5.415085</td>
</tr>
</tbody>
</table>

Variations on growth-decay models.

A. Fishery model. Same population model.
\( n(t) \): Fish population at time \( t \).
\[
\frac{dn}{dt} = -\lambda n \quad \lambda = \text{attrition rate for fish}
\]
\[
\lambda = \alpha + \beta, \quad \alpha = \text{natural mortality rate} \quad \beta = \text{fishing mortality rate}
\]
At \( t = 0 \), \( n(t) = n_0 \)
\[
\text{Then } n(t) = n_0 e^{-\lambda t}
\]

Total catch up to time \( t \),
\[
C(t) = \frac{\beta}{\lambda} (n_0 - n)
\]
When more than one variety of fish active \( \beta = \sum_j \beta_j \)
And, the volume of \( j \)th catch would be
\[
C_j(t) = \frac{\beta_j}{\lambda} (n_0 - n)
\]

B. Romantic model by J. C. Sprott
(http://sprott.physics.wisc.edu/lectures/love&hap)

\( R = \) Romeo’s love (or hate) for Juliet
\( J = \) Juliet’s love (or hate) for Romeo.
\[
\begin{align*}
\frac{dR}{dt} &= aR + bJ \\
\frac{dJ}{dt} &= cR + dJ
\end{align*}
\]
with \(a, b, c, d\) as constants.

Some “romantic” nuances:

- \(a = 0\) (Out of touch with own “feelings”)
- \(b = 0\) (Ignorant of others’ feelings)
- \(a > 0, b > 0\) (Very romantic)
- \(a > 0, b < 0\) (Narcissistic nerd)
- \(a < 0, b > 0\) (Cautious lover)
- \(a < 0, b < 0\) (As a hermit)

Therefore, \(6 \times 6\) possible combinations of the variables leading to a rich tapestry of dynamics!

Some special cases.

\[
\begin{align*}
\frac{dR}{dt} &= bJ \\
\frac{dJ}{dt} &= cR
\end{align*}
\]

\(b > 0, c > 0\) (Explosive love)
\[ b > 0, c < 0 \] (Never-ending dance?)

\[ b < 0, c > 0 \] (Never-ending dance?)
\[ b < 0, c < 0 \] (Unfinished Nerd-symphony)

(As Red comes closer, Green moves away from it which makes the Red to fly in the opposite direction, etc.)

C. Sales with advertising
When advertising stops, sales decay exponentially. There is some $t > \tau$ when this occurs.

Sales has a saturation point $M$. i.e. sales cannot increase beyond $M$.

Advertising level is $A$.

Model: \[
\frac{dS}{dt} = A \left( \frac{M - S}{M} \right) - \lambda S, \quad A(t) = A \text{ when } t < \tau > \tau, \text{ otherwise}
\]

If $X = \text{total advertising expenditure} = A \tau$

Then, \[
\frac{dS}{dt} = -bS + C, \quad \text{where } b = \left( \frac{X}{\tau M} + \lambda \right), \quad C = \frac{X}{\tau}
\]

Solution: \[
S(t) = S_0 e^{-bt} + \frac{C}{b} (1 - e^{-bt})
\]
Typical policy: Advertising policy ought to be such that would push $\mathcal{T}$ toward right but at the same time reaches $S(\tau)$ quickly. Probably, a big initial push and then followed by a steady advertising pulses with a certain periodicity. e.g.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{advertising_policy.png}
\caption{Typical advertising policy}
\end{figure}

Economic model.

A pricing model. What would be the market price $p$ for your commodity? How stable is it?

Two forces shaping the market price: Supply of commodity $S$, Demand for that commodity $D$.

General observations:

1. If $D$ goes higher, its price would push higher.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{demand_price.png}
\caption{Demand-price relationship}
\end{figure}

2. If $S$ goes higher, its price would come down.
Combining the two, we get the integrated scenario

How do we realize this equilibrium market driven price $p^*$? By adjusting ourselves to the forces of economy!

In our case, we force the process to align itself to market economy by

$$\text{price\_push} \rightarrow \text{correct} \rightarrow \text{price\_push} \rightarrow \text{correct} \ldots$$
On the other hand, compare the same phenomenon with different relative slopes as shown below

Therefore, our question comes down to this: Under what circumstances does the pricing model yield a stable price?

Suppose, \( S = s_0 - m_s p \) and \( D = d_0 + m_d p \)

Then, \( S = s_0 - \frac{m_s}{m_d} D + \frac{m_s d_0}{m_d} \)

Therefore, as long as the relative change of \( S \) with respect to \( D \) is (the absolute magnitude)
\[ \frac{dS}{dD} = \frac{m_s}{m_d} < 1 \] the cobweb will be convergent to a stable price index \( p^* \).

Think about its expansion incorporating non-linear structures.

**Guerrilla warfare.** Two different groups: Good guys, bad guys

Defenders: \( m(t) \) Return fire blindly

Guerrillas: \( n(t) \) Fire defenders having the later in full view

A typical model may be:

\[
\frac{dn}{dt} = -\alpha mn \\
\frac{dm}{dt} = -\beta n
\]

This yields \( \frac{dn}{dm} = \frac{\alpha m}{\beta} \rightarrow \int\beta dn = \alpha \int m dm \)

i.e. \( \beta(n - N) = \frac{\alpha}{2}(m^2 - M^2) \)

There is a likely draw if \( 2\beta N = \alpha M^2 \) or

\[
M = \sqrt{\frac{2\beta N}{\alpha}}
\]

Vietnam data: \( \frac{\alpha}{2\beta} < 0.001 \). Guerrillas can win even if vastly outnumbered provided both sides are subdivided into small pockets
of engagements and guerrillas attack with local numerical superiority.

Question: Do we need satellite surveillance for land warfare? (http://www.bepress.com/peps/vol8/iss1/3/)

Lancaster War model. A coupled growth-decay system.

Two warring populations.

\( n(t) \) : number of blue forces at time \( t \)
\( m(t) \) : number of red forces at time \( t \)

\[
\frac{dn}{dt} = -\alpha m \\
\frac{dm}{dt} = -\beta n
\]

\( \Rightarrow \frac{dn}{dm} = \frac{\alpha m}{\beta n} \).

Initial populations \( N \) and \( M \), respectively.

Solution: \( \alpha(M^2 - m^2) = \beta(N^2 - n^2) \)

Forces fight to draw if \( \alpha M^2 = \beta N^2 \)

This suggests that if the Blue system is 4 times as effective as the red (weapons, environment, doctrine, etc.), Red will need twice the initial force to seize a draw.

\[
\text{Also, } \frac{d}{dt}\left(\frac{dn}{dt}\right) = -\alpha \frac{dm}{dt} = \alpha\beta n \Rightarrow \frac{d^2n}{dt^2} = \alpha\beta n
\]

Its general solution would be
\[ n(t) = N \cosh \sqrt{\alpha \beta} \ t - \frac{\beta}{\alpha} N \sinh \sqrt{\alpha \beta} \ t \]
similarly,
\[ m(t) = M \cosh \sqrt{\alpha \beta} \ t - \frac{\alpha}{\beta} M \sinh \sqrt{\alpha \beta} \ t \]

A MATLAB run shows the profile of two functions against time for a given set of initial values.

Another coupled system. Lotka-Volterra equations.

Prey-Predator model.

\[ \frac{dm}{dt} = \lambda_m (1 - \alpha n) m \quad \text{Prey (host)} \]

\[ \frac{dn}{dt} = -\lambda_n (1 - \beta m) n \quad \text{Predator (parasite)} \]

Higher the parasite population, lower is the host growth rate.
Higher the host population, higher is the parasite growth rate.

Structurally, they are related in this manner
This is structurally a stable arrangement. Similar, structural approach:

A positive feedback loop. Unstable.

A negative feedback system. Product of the signs on the arcs is negative. This system is stable. Typical thermostat System.
For our predator-prey problem,

System is in equilibrium when \( \frac{dm}{dt} = 0 \) and \( \frac{dn}{dt} = 0 \)

Let equilibrium populations be \( m_e \) and \( n_e \), respectively.

Then
\[ \alpha n_e = 1 \quad \text{and} \quad \beta m_e = 1. \]

General system profile in terms of equilibrium volumes is

\[
\frac{dm}{dt} = \lambda_m \left(1 - \frac{n}{n_e}\right) m \quad \text{and} \quad \frac{dn}{dt} = -\lambda_n \left(1 - \frac{m}{m_e}\right) n
\]

from these two, we get

\[
\frac{dm}{dn} = -\frac{\lambda_m}{\lambda_n} \frac{1 - \frac{n}{n_e}}{1 - \frac{m}{m_e}} m
\]
from which we get contour solutions on the m-n plane

\[
\frac{1}{\lambda_m} \left( \ln m - \frac{m}{m_e} \right) + \frac{1}{\lambda_n} \left( \ln n - \frac{n}{n_e} \right) = \text{const}
\]

**Logistic growth model**

\[
\frac{dp}{dt} = kp \left(1 - \frac{p}{m}\right).
\]

Note. Two equilibrium points: \( p = 0 \) and \( p = m \). The equation is clearly separable. Also note that

\[
\frac{1}{p(m - p)} = \frac{1}{m} \left( \frac{1}{p} + \frac{1}{m - p} \right)
\]

Therefore, the solution comes out to be

\[
p = \frac{mCe^{kt}}{m + Ce^{kt}} \quad \text{with the constant } C = \frac{mp_0}{m - p_0}
\]

i.e.
\[ p = \frac{mp_0}{p_0 + (m - p_0)e^{-kt}} \]

At \( t \to \infty \), \( p = p_\infty \to m \). From such a reference point,

\[ p = \frac{p_\infty p_0}{p_0 + (p_\infty - p_0)e^{-kt}} \]

A **meta-stable population model**. More complex than logistic model.

\[ \frac{dp}{dt} = ap(b - p)(p - c) \] with \( a, b, c \) constants. This is often demonstrated in “tipping” behavior in a restaurant.

**Boom/Bust dynamics**

Consider a population existing on a non-renewable resource.

\[ \frac{dx}{dt} = caxy - dx \quad (x \text{ is population}) \]
\[ \frac{dy}{dt} = -axy \quad (y \text{ is resource}) \]

Dynamics of this model appears to be
What if we allow the resource to be renewable? We now have **Sustained oscillations with BB economy.** Our equations might appear as

\[ \frac{dx}{dt} = caxy - dx \quad \text{and} \]

\[ \frac{dy}{dt} = -axy + by \]

We will repeat boom/bust phenomenon cyclically over time. Cf Lotka-Volterra model of predator/prey.

4. Geopolitics of war.

A focal state is the state of our investigation. Suppose it has an area \( A \). More the area more revenue \( R \) via taxation, more recruitment for military to wage wars \( W \) which would earn more area. Only three variables: \( A, W, R \).
\[
\frac{dA}{dt} = c_1 W \quad R = c_2 A \quad \text{and} \quad W = c_3 R - c_4
\]

There may be some adversaries and peaceniks always wanting to thwart war effort; hence a \( c_4 \) with a negative sign. If earned revenue is below \( c_4 / c_3 \) war would be negatively appearing with revenue. Then

\[
\frac{dA}{dt} = c_1 c_2 c_3 A - c_1 c_4 = cA - a
\]

The equilibrium point is \( A_0 = \frac{a}{c} \). If \( A \) starts below \( A_0 \) it will disappear in time. If it starts above \( A_0 \), it will increase exponentially.
Not much interesting solution. How about introducing the notion of distance into it?

♦ Larger the area more difficult it is to control its periphery.
♦ Assume area to be a circle with radius $r$. Therefore when $A$ increases it enforces increase of its logical distance from center as $\sqrt{A}$.
♦ Assume increasing logical distance corresponds to decay in population exponentially.

Then a reasonable model could be asserted as:

$$\frac{dA}{dt} = cAe^{-\frac{\sqrt{A}}{h}} - a = cA(1 - \frac{\sqrt{A}}{h} + ...)-a$$

yielding additional equilibrium points $A_1$ and $A_2$. 
If $A$ is below $A_1$ it will disappear in time; if it is above $A_2$ in the neighborhood of $A_2$ it’d be declining, but below $A_2$ it’d be increasing. $A_2$ is thus a stable equilibrium point.

What if the area of the focal state is a linear strip having different countries at the side? Since the space one dimensional, the degree of influence away from center is inversely proportional to distance leading to an equation like

\[
\frac{dA}{dt}
\]
Collective solidarity inclusion. Assume $S$ is solidarity factor varying between $0$ and $1$. If $0$, the society lives as bunch of islands with little communication between them. At the other extreme, society lives as one national unit. Then a reasonable structure would be to suggest

$$\frac{dA}{dt} = cSA(1 - \frac{A}{h}) - a$$

Bust how does $S$ change if it’s not fixed? Let us assume a model for $S$

Collective solidarity = living with neighbors in empathy, helping each other, standing out for each other is minimal near the center of power, maximum at the periphery.

Then

$$\frac{dS}{dt} = c_2\left(\frac{A}{A_0} - 1\right)S(1 - S)$$

with $0 \leq S \leq 1$
If $A < A_0$ (too near to “Washington”, $S$ declines)
If $A > A_0$ $S$ increases. $S(1 - S)$ maximizes at 0.5

In this case, our model is a pair of equations:

\[
\begin{align*}
\frac{dA}{dt} &= c_1 SA(1 - \frac{A}{h}) - a \\
\frac{dS}{dt} &= c_2 \left( \frac{A}{A_0} - 1 \right) S(1 - S)
\end{align*}
\]